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Continued fractions connecting number theory, dynamical systems, and hyperbolic geometry

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1.1 Biography/Career Trajectory

1.1.1 How I Got Here

When I was 16, I decided to attend Johns Hopkin's Center for Talented Youth (CTY). I knew I was good at math, so I signed up for Number Theory. Suddenly, a world of formulas was transformed into proofs. Figuring out where mathematics comes from and why it works was way more exciting than manipulating algorithms. That early introduction to proofs inspired me to go into math. It was also my first introduction to math summer programs, which would go on to tremendously influence my graduate school path.

The Carleton College Summer Math Program for Women (SMP) was the other summer program that influenced my graduate school path. I started college at a tiny liberal arts school. Because I had the opportunity to double up my math classes in high school, I was immediately in more advanced classes with fewer than 10 people. One professor suggested I apply for SMP. The courses looked interesting, and being paid to study math was exciting, so I applied. Because I hadn't had a great experience at an all girls school, I wasn't initially thrilled that the program was for women. I am grateful that SMP showed me the value of an all women program. Mentoring is one of the most important ways that students are supported through school, from kindergarten through graduate school. Women are less likely to be mentored naturally, and programs like SMP step in to fill that gap. I am grateful for all of the support and advice the SMP community has given me through grad school, especially for supporting my way of being a mathematician.

Between my experiences at CTY and SMP, I knew the value of math programs outside of the standard curriculum. There were many reasons I chose the University of Illinois for graduate school. I was worried that I would struggle to find an advisor I worked well with in a department with fewer professors. I also knew the U of I ran many outreach activities in the local community. My first semester of grad school I began helping with our Association of Women in Mathematics (AWM) outreach programs. Both semesters, we hosted a series of two hour activities for middle school girls, and an all day program for high school girls.

That winter at the Joint Math Meetings, I was talking to a group of Illinois grad students about our department's outreach efforts, and we realized we didn't run any programs in the summer. So Melinda Lanius, another SMP alum, and I decided to change that. We won a $1500 grant from the University, convinced the Department of Mathematics to pay for lunch, wrote lesson plans, and found eight other grad students to help as classroom assistants. I even convinced a professor to give me a one month assistantship for the work. Through some whirlwind marketing, we managed to receive 60 applications from local high school students interested in doing math for a week in the summer. And then, the first week of August 2015, we ran the first Summer Illinois Math (SIM) Camp. In the seven months between the Joint Meetings and Camp, we didn't have time to stop and think about the impact of what we were doing. I'm pretty sure if we did, I would have been overwhelmed by imposter syndrome. That first summer, I taught a number theory and cryptography course based on my CTY textbook, and Melinda taught low dimensional topology based on our SMP notes. We modeled our courses on those that had influenced us the most, because we wanted others to have those opportunities. Summer 2018 was the fourth year of SIM Camp, and our first summer offering three camps.

SIM Camp remains almost entirely graduate student run, with undergraduate assistant instructors. As of 2018, eleven graduate students have been instructors, and four of those are SMP alums.

1.1.2 How I Found My Math

Mathematics PhD programs train students to be mathematicians, but there are many ways to be a mathematician. My way involves getting others excited about math through teaching and outreach, but it also involves research. When I started graduate school, I was not sure what area of mathematics I wanted to study.

After taking Lie theory, an area of algebra, at SMP, I really wanted to learn more about it. I took a Lie theory course my first semester of grad school, and despite spending most of the time feeling very confused, I really enjoyed it and the professor. After seeing him give a talk that I really liked, I asked him to do a reading course, to see if I was interested in working with him. I also applied to be the graduate student mentor on an undergraduate research project that he was leading. At the end of that semester, we sat down to talk about my plans, and I told him that everything that we were working on was too analytic for my tastes. We made a list of what type of math I wanted to do and who I should email about reading courses.

Ultimately, I ended up with a professor who does even more analysis, but I still don't. At least, all of the analysis I do is either handled by the geometry or only uses integrals that are easier than some of our calc 2 assignments. But we have similar approaches to working on math, even if we have different approaches to the actual problem. Sometimes we have two hour meetings and really work things out, and sometimes we meet for five minutes because we have nothing to say. Anyone who is worried about finding an advisor or thesis problem should talk to a professor or older student they like, either mathematically or personally. Talking to someone with a better feel of the mathematical landscape is incredibly helpful. But it's also important to know it's ok to do something different than your advisor. Having different approaches to problems can be incredibly useful.

My research is between a few areas of mathematics, which I like, because it allows me to work with lots of different people. Sometimes I'm a number theorist, sometimes I'm a geometer, frequently I'm a dynamicist and ergodic theorist. Really, I study continued fractions (number theory) by drawing pictures (two dimensional geometry) and using those pictures to figure out what happens when you apply the same map over and over (dynamics).

1.2 Abstract

Abstract: Continued fractions are a way to represent numbers as $a_0 + \cfrac{e_1}{a_1 + \cfrac{e_2}{\dots}}$. They appear in various areas of mathematics, from number theory to dynamical systems to geometry. I will describe functions which generate these fractions. The continued fraction expansions provide a nice description of paths on the geometric surfaces. The geometric properties of these pictures help us to describe patterns in the continued fraction expansions, and the continued fraction expansions provide a compact description of the geometry.

Various types of continued fractions correspond to different rules for the numerators and denominators The first example is the "regular" or "simple" continued fractions, where all of the numerators, e_i, are 1. I then show how the dynamics and geometry change when all of the numerators are ± 1 and the denominators, a_i, are all even.

Mathematical background: Since I am using functions to generate continued fractions, you should be comfortable with function composition, as well as preimages of points. I will refer to functions that are onto (every point has a preimage) and one-to-one (different points have different images) as invertible. There is also a lot of adding and multiplying fractions and use of the fact that $\frac{1}{\frac{1}{a}} = a$.

For the geometry, you should be comfortable with rotating objects in space. Complex numbers will also show up throughout. If you are unfamiliar with complex variables, you can think of the point $z = x + iy$ as (x, y) in the plane. Circles of radius r centered at $x + iy$ in the complex plane can be written as $x + iy + re^{it}$ where $t \in (0, 2\pi)$. Here, all of our circles are centered on the x-axis, so $y = 0$.

1.3 Introduction

1.3.1 What is a continued fraction?

Continued fractions are a way to represent real numbers as

$$x = a_0 + \cfrac{e_1}{a_1 + \cfrac{e_2}{a_2 + \cfrac{e_3}{a_3 + \dots}}},$$

with various restrictions on the e_i and a_i. Typically, the numerators e_i are ± 1 and the denominators a_i are positive integers, but there could be further restrictions. For example, you may have heard that π is approximately $\frac{22}{7}$, which comes from the continued fraction approximation $\pi \approx 3 + \frac{1}{7} = 3.\overline{142857}$. In fact,

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \dots}} \tag{1.1}$$

where $3 + \frac{1}{7+\frac{1}{15}}$ agrees with π to four decimal places. In fact, for almost all numbers, the continued fraction converges faster than the decimal expansion. We can also write π with only even denominators as

$$\pi = 4 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{8 + \cfrac{1}{2 - \dots}}}}. \tag{1.2}$$

The rules for different types of continued fraction expansions, such as "every numerator is $+1$" or "every denominator is even," connect to different geometry and matrix groups. This paper will explain how to connect geometry to the regular continued fractions, where every numerator is $+1$, and the continued fractions with even denominators.

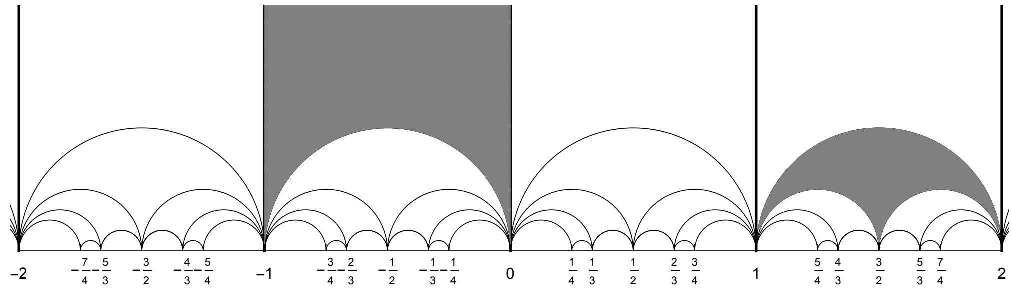

Figure 1.1. The image used for the geometric description in Section 1.5.2. The two hyperbolic triangles with vertices $\{-1, 0, \infty\}$ and vertices $\{1, \frac{3}{2}, 2\}$ are highlighted in grey.

1.3.2 Preview of connections with other fields of mathematics

The continued fraction expansions provide a nice description of paths on the geometric surfaces. As part of the geometric construction, I introduce the hyperbolic plane and describe a tiling made of hyperbolic triangles in Section 1.5. The *regular continued fraction expansions* are like the continued fraction expansion of π in (1.1), where every numerator is $+1$ and every denominator is a positive integer. These continued fractions correspond to something geometers study called a *modular surface*, which is like a sphere where you stretch one point out to infinity, and pinch two other points down like the point of a cone. This shape comes from folding the *hyperbolic triangles* in Figure 1.1 into thirds in a specific way to get a quadrilateral, then gluing adjacent sides together. If you did a similar gluing with a square piece of paper, you would fold along the diagonal to glue the left side to the bottom and the right side to the top, then inflate the resulting shape. You would then have cone points at the 3 remaining corners.

Both the modular surface and the regular continued fractions are related to the group of 2×2 matrices with integer entries and determinant 1. This relation allows us to construct the modular surface, as well as use the regular continued fractions to describe something called *group equivalences*. The *cutting sequence construction for regular continued fractions* in Section 1.5.2 let us describe which "straight lines" on the modular surface come back to their starting point, like the equator or longitude lines on a sphere (for example Series's paper "The modular surface and continued fractions" [11]). Facts about the *invariant measure for the regular continued fractions* also prove that there are lines that hit almost every point on the surface, but never intersect themselves. My goal in this paper is explaining the geometric construction behind the proofs of these facts, rather than prove them.

The continued fraction expansion of π in (1.2) is the *even continued fraction expansion*, where every numerator is ± 1 and every denominator is a positive even number. My work connects the even continued fractions to a different modular surface. To get this surface you fold the hyperbolic triangle in half through one of the vertices, which also folds one of the sides in half. Glue the folded side to itself and the other two side to each other. This gives a sphere with two points dragged to infinity and one cone point. This modular surface is related to determinant 1 matrices of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a, b, c, d are integers and ac and bd are always even. This is a subgroup of the group from the regular continued fractions. Now, the even continued fractions describe group equivalences and straight lines on the surface. Because the invariant measure for the even continued fractions described in Proposition 4 can be infinite, it is far more difficult to prove the types of results that tell us there exist lines that visit almost every point without intersecting. However, this fact is already known for modular surfaces related to subgroups of matrices with determinant 1. One of the main ways people prove similar results is by finding a geometric description.

In Section 1.4, I describe regular, even, and extended continued fractions as *dynamical systems*. The dynamical systems are what allow us to translate between continued fractions and geometry. I introduce the hyperbolic plane at the beginning of Section 1.5, then define a *tiling* of the hyperbolic plane in 1.5.1, Finally, I connect the regular continued fraction expansions to the hyperbolic plane using *cutting sequences* in Section 1.5.2. I then repeat this process for the even continued fractions in Section 1.6. For both the regular and even continued fractions, I describe the continued fraction expansion of two irrational numbers based on how a semicircle connecting these points crosses the tiling from Section 1.5.1. In fact, one part of the dynamical system is what we call an *invariant measure*, which I find using the geometric construction. My goal is to explain why you can view continued fractions and paths on these geometric surfaces as essentially the same thing. That is, I can give you continued fractions and you give me the cutting sequence, or I can give you the cutting sequence and you give me the continued fraction.

1.3.3 Computing continued fractions

To get the *regular continued fraction expansion* of a number $x > 0$, we start with the largest integer less than x and call this integer a_0. Now we can write $x = a_0 + \frac{1}{r_1}$ for some $r_1 > 1$. To get the next digit, repeat this process for r_1, so a_1 is the largest integer less than r_1 and $x = a_0 + \frac{1}{a_1 + \frac{1}{r_2}}$. If we keep going, we get

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} = [a_0; a_1, a_2, \dots].$$

It will be helpful to use a running example throughout the paper. I will use α, an irrational number very close to $\frac{36}{13}$. First, let's find the regular continued fraction expansion of $\frac{36}{13}$. The first step of finding the regular continued fraction expansion of $\frac{36}{13}$ is to write it as $2 + \frac{10}{13} = 2 + \frac{1}{13/10}$. To find the next digit, we rewrite $\frac{13}{10}$ as $1 + \frac{3}{10} = 1 + \frac{1}{10/3}$. Thus, we get $2 + \frac{1}{1+\frac{1}{10/3}}$. Finally, we rewrite $\frac{10}{3}$ as $3 + \frac{1}{3}$. Thus, we find that $\frac{36}{13} = 2 + \frac{1}{1+\frac{1}{3+\frac{1}{3}}}$. To define α, we replace the last 3 with $3 + z$ for $z \in [0, 1)$, so

$$\alpha := 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3 + z.}}} \tag{1.3}$$

When we need an example in $[0, 1)$, I will use $1/\alpha$.

To get the regular continued fraction expansion of negative numbers, we find the continued fraction expansion of the absolute value, then write $x = -[a_0; a_1, a_2, \dots]$. The regular continued fraction expansion terminates exactly when x is rational, like how the decimal expansion of rational numbers terminate or repeat (see for example, Chapter 3, Lemmas 3.1 and 3.2 of *Ergodic Theory with a view towards Number Theory* [6]). The regular continued fraction expansions of irrational numbers are unique. However, the regular continued fraction expansion of rational numbers is not unique because the final denominator could be $n = n - 1 + \frac{1}{1}$. We require that the last denominator is greater than 1 to get a unique continued fraction expansion. The ambiguity of continued fraction expansions is similar to how $1 = 0.99\dots$, but we typically write the finite decimal expansion. Finally, it is useful to notice that

$$\frac{1}{[a_0; a_1, a_2, \dots]} = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}} = \frac{1}{a_0 + [0; a_1, a_2, \dots]} = [0; a_0, a_1, a_2, \dots] \tag{1.4}$$

$$\frac{1}{[0; a_1, a_2, \dots]} = \cfrac{1}{\cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}} = a_1 + \frac{1}{a_2 + [0; a_3, \dots]} = [a_1; a_2, a_3, \dots]. \tag{1.5}$$

Notice that if $x \in \left(\frac{1}{k+1}, \frac{1}{k}\right]$, then $x = \dfrac{1}{k+z}$ for some $z \in [0, 1)$. So, we can break up $(0, 1)$ based on the first digit of the regular continued fraction expansion. In Section 1.4, I define a function, or *map*, that uses this partition of $(0, 1)$ to generate continued fraction expansions.

1.4 Dynamical systems

Dynamicists study what happens when we apply the same map over and over. In general, a *dynamical system* consists of a set X, a surjective map (function) $T : X \to X$, and a function μ : subsets of $X \to \mathbb{R}_{\geq 0}$ that measures the size of sets[1]. We call μ a measure and require that for disjoint sets, the size of the union of the sets equals to sum of the sizes

[1]Dynamical systems also involve a σ-algebra which you can think of as subsets of X.

of the (individual) sets. In this paper, all of the measures are integrals of functions. For example, the standard measure on $\mathbb{R}$ is the length of an interval. As an integral, this measure is $\mu([a,b]) = \int_a^b dx = b - a$. Since we are interested in how points $x \in X$ move under T, we say T *acts* on X.

1.4.1 Regular continued fractions

We can use a dynamical system to generate the regular continued fractions. Here, $X = [0,1]$, and $T : [0,1] \to [0,1]$ (Figure 1.2) is given by

$$T(x) = \begin{cases} \frac{1}{x} - k & \text{for } x \in \left(\frac{1}{k+1}, \frac{1}{k}\right] \\ 0 & \text{for } x = 0. \end{cases}$$

We call T the *Gauss map for the regular continued fractions*. The first digit after 0 of the continued fraction expansion

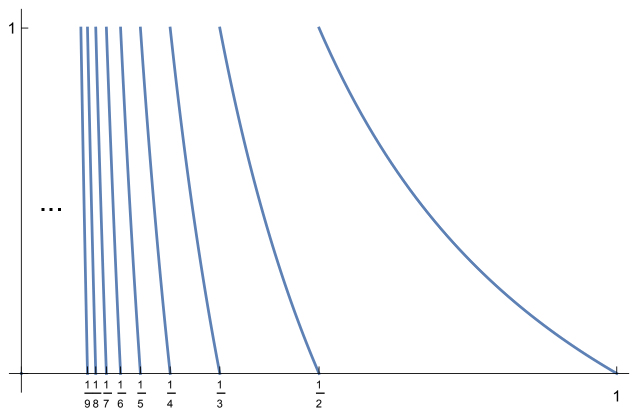

Figure 1.2. Gauss Map T

of x is k. If we look at what T does to the continued fraction expansion of $x = [0; a_1, a_2, \dots]$, we get

$$T([0; a_1, a_2, \dots]) = a_1 + \frac{1}{a_2 + \dots} - a_1 = \frac{1}{a_2 + \dots} = [0; a_2, a_3, \dots].$$

So, T deletes the first digit of the continued fraction expansion. In fact, we can write the regular continued fraction expansion of x by repeatedly applying T and recording which integer we subtract.

Using α from (1.3), we see that $\frac{1}{3} < \frac{1}{\alpha} \approx \frac{13}{36} \leq \frac{1}{2}$. Thus, $T(\frac{1}{\alpha}) = \frac{1}{1/\alpha} - 2$ and the first digit of the continued fraction expansion of $\frac{1}{\alpha}$ is 2. Using the fact that $\alpha - 2 \approx \frac{36}{13} - 2 = \frac{10}{13}$, we see that $\frac{1}{2} < T(\frac{1}{\alpha})) \leq 1$, so $T(T(\frac{1}{\alpha}) = \frac{1}{T(1/\alpha)} - 1$. Thus, the second digit of the continued fraction expansion of $\frac{1}{\alpha}$ is 1. We can continue this process to find all of the digits of $\frac{1}{\alpha}$. Recall from (1.5) that the continued fraction expansion of $\frac{1}{\alpha}$ also tells us the continued fraction expansion of α. Thus, we can find the continued fraction expansion of all positive real numbers using a dynamical system. For negative numbers, we find the continued fraction expansion of the absolute value.

The last part of the dynamical system is the measure. We want a *T-invariant measure*, which is a function $d\mu = h(x)dx$ so that for any $(a,b) \subseteq [0,1]$,

$$\int_{T^{-1}(a)}^{T^{-1}(b)} d\mu = \int_a^b d\mu.$$

That is, any set is the same size as its preimage. Since the size of the set does not change, we can more easily study how T moves subsets of $[0,1]$.

Theorem 1. *[Gauss] The probability measure* $\mu([a,b]) = \dfrac{1}{\log 2}\displaystyle\int_a^b \frac{dx}{1+x}$ *is* T*-invariant.*

Proof. A full proof is in chapter 3 of *Ergodic theory with a view towards Number Theory* [6], but try proving it for yourself when $a = 0$. First, we find $T^{-1}(0)$ and $T^{-1}(b)$ for a fixed k

$$\begin{aligned} 0 &= \frac{1}{x} - k \quad so \quad \frac{1}{k} = x, \\ b &= \frac{1}{x} - k \quad so \quad \frac{1}{b+k} = x. \end{aligned}$$

Thus,

$$T^{-1}([0,b]) = \bigcup_{k=1}^{\infty} \left[\frac{1}{b+k}, \frac{1}{k}\right].$$

We also need that

$$\int_{T^{-1}([0,b])} \frac{dx}{1+x} = \sum_{k=1}^{\infty} \int_{\frac{1}{b+k}}^{\frac{1}{k}} \frac{dx}{1+x}.$$

Now, try to finish the proof yourself. □

There is a no general process for finding the T-invariant measure. Section 1.5.2 will walk through a geometric way to find an invariant measure for the continued fractions. It can be easier for invertible maps, so we are going to define a two dimensional, invertible version of the Gauss map T, called the *natural extension* $\bar{T}$, on $[0,1]^2$.

The natural extension $\overline{T}$ will move the first digit of x to the first digit of y. That is, if $(x,y) \in [0,1]^2$, and

$$(x,y) = \left(\cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}, \cfrac{1}{b_1 + \cfrac{1}{b_2 + \dots}} \right), \text{ then } \bar{T}(x,y) = \left(\cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}, \cfrac{1}{a_1 + \cfrac{1}{b_1 + \dots}} \right).$$

Formally, we define $\bar{T} : [0,1]^2 \to [0,1]^2$ by

$$\bar{T}(x,y) = \begin{cases} \left(\frac{1}{x} - k, \frac{1}{k+y}\right) \text{ for } x \in \left(\frac{1}{k+1}, \frac{1}{k}\right] \\ (0,y) \text{ for } x = 0. \end{cases}$$

The first coordinate of $\bar{T}$ is the Gauss map, and the second is the inverse of the Gauss map for a given k.

I will describe the regular continued fractions geometrically in Section 1.5.2. First, I am going to describe continued fractions with even denominators. The geometric descriptions of the other types of continued fraction expansions are in Section 1.6.

1.4.2 Even continued fractions

We do not need to require that all numerators are $+1$ to get a valid continued fraction expansion. In this section, we will let our numerators be ± 1, but require that the denominators are all even, which were first introduced by Schweiger [10].

Returning to our example of $\frac{36}{13}$, the first step in finding the even continued fraction expansion is the same, since $2 + \frac{10}{13} = 2 + \frac{1}{13/10}$. However, we cannot write $\frac{13}{10} = 1 + \frac{1}{10/3}$, since 1 is odd. Instead, we round *up*, getting $\frac{13}{10} = 2 - \frac{7}{10} = 2 - \frac{1}{10/7}$. We repeat this for $\frac{10}{7} = 2 - \frac{1}{7/4}$ and $\frac{7}{4} = 2 - \frac{1}{4}$. Putting this all together, we get

$$\frac{36}{13} = 2 + \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{4}}}} = [\![(2,+1);(2,-1)(2,-1)(2,-1)(2,-1)(4,+1)]\!].$$

The $(2,+1)$ indicates that the 2 was an *underestimate* and we add the remainder, while the $(2,-1)$ indicated the 2 is an *overerestimate* and we subtract the remainder. Since we determine this information at the same time, we group the denominator with the *next* numerator. In general, the even continued fraction expansion of real numbers greater than 1 is

$$a_0 + \frac{e_0}{a_1 + \ldots} = [\![(a_0, e_0); (a_1, e_1)(a_2, e_2)(a_3, e_3)\ldots]\!],$$

where $e_i = \pm 1$, and every a_i is a positive even integer. We write real numbers $x \in (0, 1]$ as

$$x = \cfrac{1}{a_1 + \cfrac{e_1}{a_2 + \cfrac{e_2}{a_3 + \cdots}}} = [\![0; (a_1, e_1)(a_2, e_2)(a_3, e_3)\ldots]\!],$$

similar to our notation for the regular continued fractions.

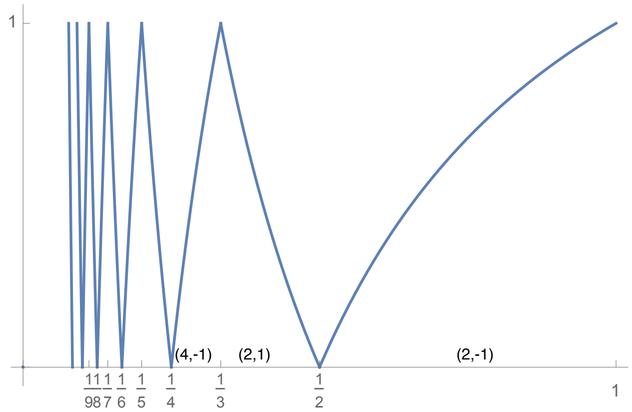

Figure 1.3. Even Gauss Map

The even Gauss map $T_e : [0, 1] \to [0, 1]$ (Figure 1.3) is given by

$$T_e(x) = \begin{cases} \dfrac{1}{x} - 2k & \text{if } x \in \left(\dfrac{1}{2k+1}, \dfrac{1}{2k}\right] \\ \dfrac{-1}{x} + 2k & \text{if } x \in \left(\dfrac{1}{2k}, \dfrac{1}{2k-1}\right] \\ 0 & \text{if } x = 0. \end{cases} \quad k \geq 1$$

If we plug in the even continued fraction expansion to T_e, we again delete the first digit (a_1, e_1) of x, i.e.

$$T_e\left([\![(a_1, e_1)(a_2, e_2)(a_3, e_3)\ldots]\!]\right) = [\![(a_2, e_2)(a_3, e_3)(a_4, e_4), \ldots]\!].$$

That is, $(a_1, e_1) = (2k, +1)$ when $x \in \left(\frac{1}{2k+1}, \frac{1}{2k}\right]$ and $(a_1, e_1) = (2k, -1)$ when $x \in \left(\frac{1}{2k}, \frac{1}{2k-1}\right]$.

One way to convert between different types of continued fraction expansions is the *insertion and singularization algorithm*, which is really just the following equation applied in a particular way:

$$a + \cfrac{\epsilon}{1 + \cfrac{1}{b + z}} = a + \epsilon + \frac{-\epsilon}{b + 1 + z}.$$

When we move from the right hand side of the equation to the left hand side, the algorithm is called *insertion*, since it inserts a 1. Moving in the other direction is called *singularization*. In her master's thesis, Masarotto describes exactly when to use insertion and when to use singularization for converting regular continued fractions to even continued

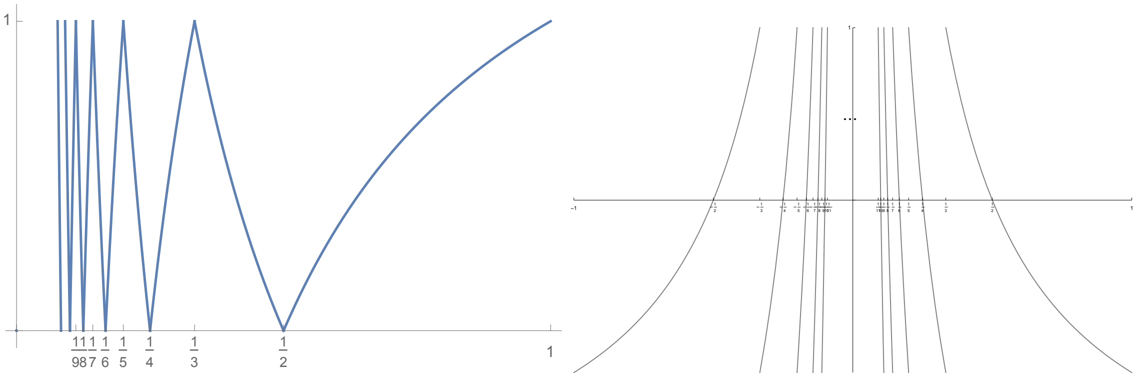

Figure 1.4. Compare the graph of the even Gauss map (left) with the extended even Gauss map (right). The even Gauss map is a "folded" version of the extended even Gauss map

fractions [9]. This algorithm allows us to find that our example $\alpha \approx \frac{13}{36}$, has continued fraction expansions

$$\alpha = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3 + z}}} = 2 + \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{4 + z}}}} = [\![(2, +1); (2, -1)(2, -1)(2, -1)(4, +1) \dots]\!]. \tag{1.6}$$

In order to construct the natural extension of the even Gauss map, we need to define the *extended even continued fractions*. [2] For $y \in [-1, 1]$, the extended even continued fraction expansion is

$$y = \cfrac{f_0}{b_0 + \cfrac{f_1}{b_1 + \cfrac{f_2}{b_2 + \cdots}}} = \langle\!\langle (f_0/b_0)(f_1/b_1)(f_2/b_2) \dots \rangle\!\rangle,$$

While we will not define the Gauss map for the extended even continued fractions here, we can look at the first digit (f_0/b_0) get a sense of the difference between the two continued fraction expansions. For the *extended even continued fractions*, $f_0 = +1$ if $y \in (0, 1]$ and $f_0 = -1$ if $y \in [-1, 0)$. Then, $b_0 = 2k_0$ for $|y| \in (\frac{1}{2k-1}, \frac{1}{2k+1})$. For the *even continued fractions,* in the above notation, $f_1 = +1$ if b_0 is an underestimate for $\frac{1}{x}$ and $f_1 = -1$ if b_0 is an overestimate for $\frac{1}{x}$. The difference in notation emphases that the method for determining, and therefore indexing, the digits is different. The difference also appears in the way the the natural extension moves the digits.

We now define $\bar{T}_e$ on $[0, 1] \times [-1, 1]$ to be

$$\bar{T}_e(x, y) = \begin{cases} \left(\frac{1}{x} - 2k, \frac{1}{2k + y}\right) & \text{if} \quad x \in \left(\frac{1}{2k+1}, \frac{1}{2k}\right] \\ \left(\frac{-1}{x} + 2k, \frac{-1}{2k + y}\right) & \text{if} \quad x \in \left(\frac{1}{2k}, \frac{1}{2k-1}\right] \\ (0, y) & \text{if} \quad x = 0. \end{cases}$$

Looking at the even continued fraction expansion of $x = [\![(a_1, e_1)(a_2, e_2), \dots]\!]$ and the extended even continued fraction expansion of $y = \langle\!\langle (f_0/b_0)(f_1/b_1), \dots \rangle\!\rangle$, we get

$$(x, y) = \left(\cfrac{1}{a_1 + \cfrac{e_1}{a_2 + \dots}}, \cfrac{f_0}{b_0 + \cfrac{f_1}{b_1 + \dots}}\right), \quad \overline{T}_e(x, y) = \left(\cfrac{1}{a_2 + \cfrac{e_2}{a_3 + \dots}}, \cfrac{e_1}{a_1 + \cfrac{f_0}{b_0 + \dots}}\right).$$

[2] It is possible to define a natural extension of the even Gauss map without the dual continued fractions, but the invariant measure then depends on whether $e_1 = +1$ or $e_1 = -1$. The geometric construction would also need to be more complicated, involving cases for $e_1 = \pm 1$. The regularly continued fractions are what we call *self dual*, meaning the dual continued fractions are still the regular continued fractions.

Thus, $\overline{T}_e$ moves the first digit of x to the first digit of y. In fact, $\overline{T}_e^{-1}$ is the natural extension of the extended even Gauss map, so we only need to study $\overline{T}_e$ to describe both the even continued fractions and the extended even continued fractions.

1.5 Hyperbolic space

In order to connect continued fractions to geometry, we use the hyperbolic plane. One way to think about it is a space with too much material around each point to flatten out, like lettuce or coral. In the usual plane, six equilateral triangles can share a vertex without gaps or overlaps. A sphere can have up to five equilateral triangles sharing a vertex, like an inflated icosahedron. The hyperbolic plane can have up to infinitely many equilateral triangles sharing a vertex without gaps or overlaps.

We need a way to draw hyperbolic space, so we define the *upper half plane* where $\mathbb{H} := \{x + iy \mid y > 0\} \cup \{\infty\}$. We call the "top" of the upper half plane ∞. In order to distinguish the upper half plane from the Euclidean plane, we have to define a new notion of distance. In $\mathbb{H}$, the shortest path between two points, called a *geodesic*, is on a semi-circle centered on the x-axis or a straight line perpendicular to the x-axis.

A *hyperbolic triangle* is a region of upper half plane bounded by three geodesics, like how a Euclidean triangle is bounded by three straight lines. We say that two vertical lines have a vertex ∞ at the "top" of the upper half plane, so form two sides of a hyperbolic triangle. When I need to refer to a specific hyperbolic triangle, I will list the vertices.

In Figure 1.5, there are three hyperbolic triangles drawn for each of the vertices $-3/2, -1/2, 1/2$, and $3/2$. There are also six hyperbolic triangles drawn for at $-2, -1, 0, 1$, and 2. The six triangles that share 0 as a vertex are: $\{-1/2, -1/3, 0\}$, $\{-1, -1/2, 0\}$, $\{-1, 0, \infty\}$, $\{0, 1, \infty\}$, $\{0, 1/2, 1\}$, and $\{1/3, 1/2, 0\}$.

1.5.1 Farey tessellation and hyperbolic geodesics

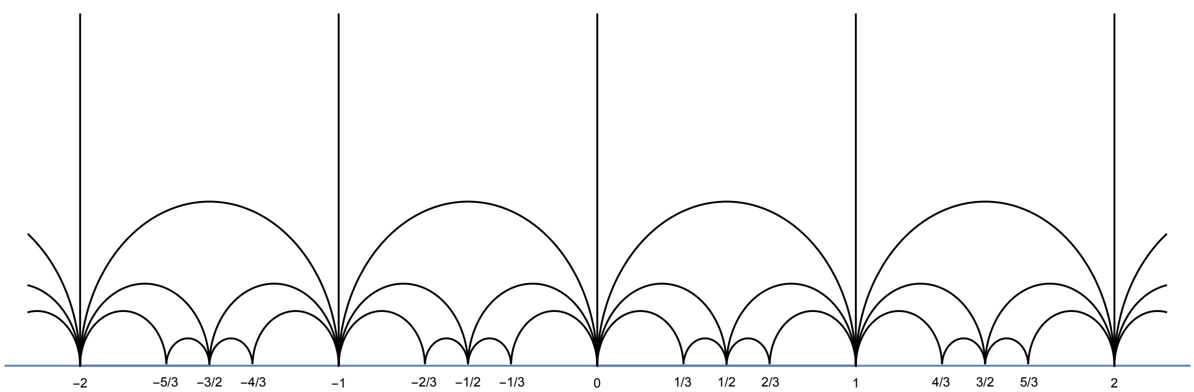

Figure 1.5. Part of the Farey tessellation with geodesics connecting rational numbers with denominators less than or equal to 3.

In order to describe continued fractions using geometry, we also need a *tessellation* or tiling of the upper half plane. In this case, each "tile" is a hyperbolic triangle with vertices either on the x-axis or at ∞. A tessellation is a rule for covering a space with tiles with no gaps or overlaps. In two dimensions, a tessellation is often called a tiling, like how tiles create patterns that cover a floor or wall without any gaps or overlaps.

To generate continued fractions using hyperbolic geometry, we use the *Farey tessellation*. First, we draw a vertical line at each integer. Next, for each n we define a set $F_n = \{\frac{p}{q} : p, q \in \mathbb{Z}, 0 < q \leq n\}$ which is the set of rational numbers with denominator less than or equal to n. We will list the numbers in order.

$$F_1 = \{\dots, -2, -1, 0, 1, 2, \dots\}$$
$$F_2 = \{\dots, -\tfrac{3}{2}, -1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots\}$$
$$F_3 = \{\dots, -1, -\tfrac{2}{3}, -\tfrac{1}{2}, -\tfrac{1}{3}, 0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, 1, \tfrac{4}{3}, \dots\}$$

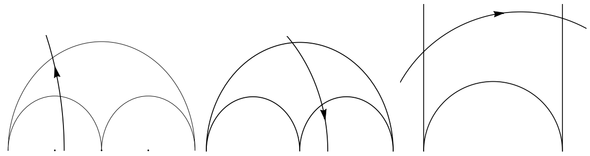

Figure 1.6. Some oriented geodesics crossing two sides of a hyperbolic triangle, which meet at a vertex on the left.

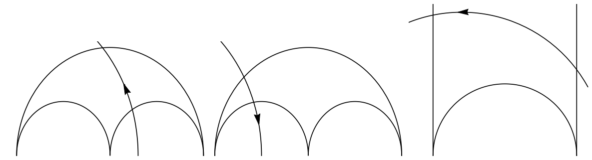

Figure 1.7. Some oriented geodesics crossing two sides of a hyperbolic triangle, which meet at a vertex on the right.

To finish drawing the Farey tessellation, we connect each number to its neighbors in each F_n using a geodesic. So for F_1, we connect each integer k to $k-1$ and $k+1$. For F_2, we connect $k+1/2$ to k and $k+1$. Figure 1.5 shows the tessellation up to F_3.

For the geometric description of continued fractions, we will consider *oriented geodesics*, which have a starting point and an ending point on the x-axis. We will walk along a geodesic γ from the starting point γ_- to the ending point γ_+ to find continued fraction expansions of irrational numbers γ_- and γ_+. This construction also works for the regular continued fraction expansion of rational numbers, but it is a bit trickier [11]. We will also define a map that acts on (γ_+, γ_-) similarly to the natural extension of the appropriate Gauss map.

1.5.2 Cutting sequences and continued fractions

The geometric description of the regular continued fractions given here is largely due to Series's paper "The modular surface and continued fractions" [11]. In addition to the argument outlined below, she uses tangent bundles, the topology of the infinite strings, and some group theory. It is a great paper if you are familiar with those topics. The next few sections outlines her argument but leaves out details in order to focus on the big ideas.

We can use an oriented geodesic to find the regular continued fraction expansion of its starting and ending points based on how it crosses tiles in the Farey tessellation. Consider an oriented geodesic γ starting at $\gamma_- \in (0,1)$ and ending at $\gamma_+ < -1$, as in Figure 1.8, or starting at $\gamma_- \in (0,1)$ and ending at $\gamma_+ < -1$ as in Figure 1.9. The geodesic crosses a sequence of tiles in the Farey tessellation, each time the crossing two sides of the hyperbolic triangle. Since γ must cross the line $x = 0$, we will label the point $\gamma \cap \{x = 0\} = u_\gamma$. The part of the geodesics between γ_- and u_γ will help us find the continued fraction expansion of γ_-, while the part of the geodesic between u_γ and γ_+ will help us find the continued fraction expansion of γ_+.

Let's start with the case $\gamma_- \in (-1, 0), \gamma_+ > 1$, as in Figure 1.8. When $\gamma_+ \in [n, n+1)$, the geodesic γ crosses the n vertical lines $x = 1, 2, \ldots, n$ after u_γ. In Figue 1.8, $\gamma_+ = \alpha \in [2,3)$, and γ crosses the vertical lines $x = 1$ and $x = 2$. We mark the point where γ crosses $x = n$ with v_γ. Another way of saying this is γ crosses n Farey cells between u_γ and v_γ. For each cell, γ crosses two edges which "meet" at ∞. If you walk along the geodesic from γ_- to γ_+, ∞ will be on your left as you cross each of the n cells between u_γ and v_γ. Thus, we label each of the n segments of γ between u_γ and v_γ with L. When $\gamma_- \in (0,1)$ amd $\gamma_+ < -1$, ∞ is on the right, so the segments are labeled R, as in Figure 1.9.

Each time γ crosses a Farey cell, it crosses two edges that meet at a vertex. If you stand on the oriented geodesic facing γ_+, this vertex will be on either the left (Figure 1.6) or the right (Figure 1.7). We will walk along the oriented geodesic in the direction indicated by the arrow and record whether the vertex is on the left or the right from the perspective of the oriented geodesic and label the corresponding sections of the geodesic L and R respectively. For example, in Figure 1.8, there are two cells between γ_- and u_γ labeled R, and each of the three edges that bound these cells meet at 0. The unlabeled segment closest to γ_- crosses two edges which meet at $\frac{-1}{2}$ on the left of the geodesic.

Since it is less clear how to get the rest of the continued fraction expansions of γ_- and γ_+, the next step is to define a map that acts like $\overline{T}$ on (γ_+, γ_-) to find the next digit. Let $\gamma_+ = sign(\gamma_+)[n_1; n_2, n_3, \ldots]$ and $\gamma_- = -sign(\gamma_+)[0; n_0, n_{-1}, n_{-2}, \ldots]$. Define

$$\rho(x,y) = \left(\frac{1}{sign(\gamma_+)n_1 - x}, \frac{1}{sign(\gamma_+)n_1 - y} \right). \tag{1.7}$$

For $\gamma_+ = \alpha$ and $\gamma_- = -1/\alpha$, $\rho(\gamma_+, \gamma_-) = \left(\frac{1}{2 - \left(2 + \frac{1}{1+\ldots}\right)}, \frac{1}{2+\frac{1}{\alpha}} \right) = \left(-\left(1 + \frac{1}{3+\ldots}\right), \frac{1}{2+\frac{1}{\alpha}} \right).$

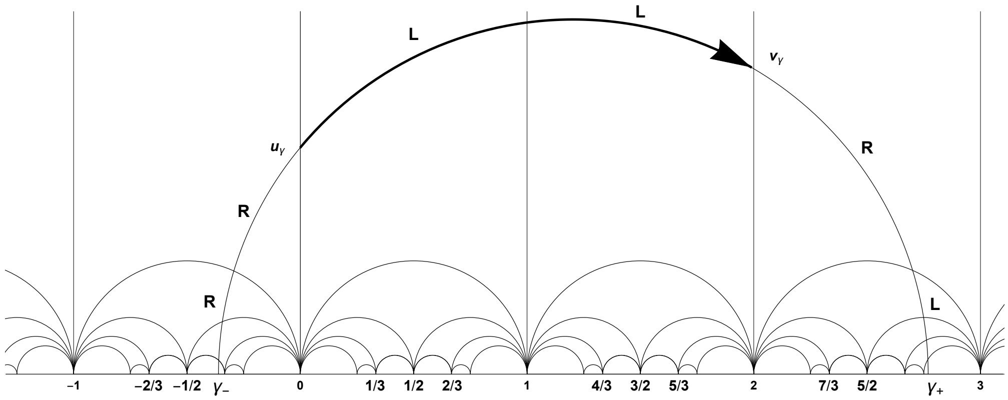

Figure 1.8. The oriented geodesic starting at $\gamma_- = -1/\alpha$ and ending at $\gamma_+ = \alpha$. Segments are labeled L or R based on how the oriented geodesic crosses the Farey tessellation, giving the cutting sequence $R^2 u_\gamma L^2 v_\gamma RL$.

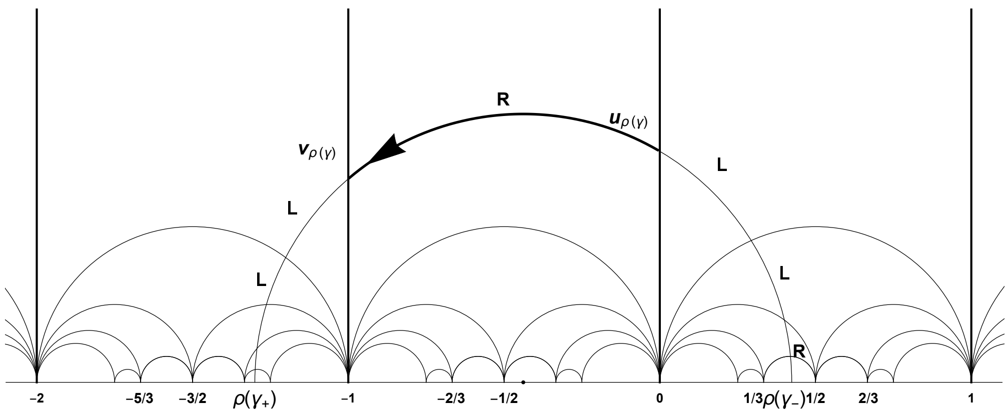

Figure 1.9. The oriented geodesic connecting $\rho(\gamma_+)$ and $\rho(\gamma_-)$ when $\gamma_+ = \alpha$ and $\gamma_- = -1/\alpha$. Segments are labeled L or R based on how the oriented geodesic crosses the Farey tessellation giving the cutting sequence $RL^2 u_{\rho(\gamma)} R v_{\rho(\gamma)} L^2$.

In general, ρ takes the first digit of γ_+ and moves it to the first digit of γ_-, just like $\overline{T}$. It also multiplies both endpoints by -1, but that does not cause any problems. By abuse of notation, I will also use $\rho(x)$ to refer to the image of one endpoint, and $\rho(\gamma)$ for the oriented geodesic connecting $\rho(\gamma_+, \gamma_-)$. The section of the cutting sequence between u_γ and v_γ describes first digit of the regular continued fraction expansion of γ_+, the section of the cutting sequence between $u_{\rho(\gamma)} = \rho(v_\gamma)$ and $v_\rho(\gamma)$ describes the second digit, and so on. We can use these ideas to find the continued fraction expansion of γ_+, like how we can use T to find the regular continued fraction expansion of a number.

Theorem 2 (Series's Theorem A [11])**.** *For the oriented geodesics described above:*

- *When* $\gamma_+ = [n_1; n_2, \dots]$ *and* $\gamma_- = -[0; n_0, n_{-1}, \dots]$*, the geodesic has cutting sequence* $\dots L^{n_{-1}} R^{n_0} u_\gamma L^{n_1} v_\gamma R^{n_2} \dots$.
- *When* $\gamma_+ = -[n_1; n_2, \dots]$ *and* $\gamma_- = [0; n_0, n_{-1}, \dots]$*, the geodesic has cutting sequence* $\dots R^{n_{-1}} L^{n_0} u_\gamma R^{n_1} v_\gamma L^{n_2} \dots$.

There are a few things to notice before sketching a proof for our example

$$\gamma_+ = [2; 1, 3, 3, \dots], \quad \gamma_- = -[0; 2, 1, 3, 3, \dots].$$

The first is that the cutting sequences of γ_+ and γ_- are independent. That is, if we take the two geodesics γ from γ_- to γ_+ and γ' from γ'_- to γ_+, the cutting sequence describing γ_+ is the same. The second thing to notice is that if one endpoint is rational, there is ambiguity in the cutting sequence, since the last letter could be L or R. This corresponds to the ambiguity of the continued fraction expansion, where $L^n R$ corresponds to $n + \frac{1}{1} = n + 1$, which in turn corresponds to L^{n+1}. The ambiguity of cutting sequences is why the rational case is trickier.

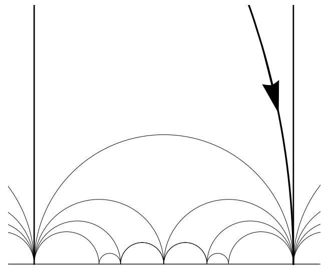

Figure 1.10. A geodesic ending at the corner n+1, which could be labeled L or R.

Proof of Theorem 2 for $\gamma_+ = \alpha$ and $\gamma_- = \frac{-1}{\alpha}$. The cutting between $x = 0$ and γ_+: Let v_γ be the last place that γ crosses a vertical line $x = n_1$. In this case, v_γ is where γ crosses the line $x = 2$. Then the bold section of γ in Figure 1.8 crosses 2 tiles between u_γ and v_γ, both labeled L. Thus, we know part of the cutting sequence is $u_\gamma L^2 v_\gamma$. Any oriented geodesic ending between 2 and 3 will cross the line $x = 2$, then turn right to cross the semicircle connecting 2 to 3. Thus, the cutting sequence will contain $u_\gamma L^2 v_\gamma$, and the first digit of the continued fraction expansion is 2.

To find the next part of the sequence, we need to look at $\rho(\gamma_+, \gamma_-)$ since ρ deletes the first digit of the regular continued fraction expansion of γ_+ and move it to γ_-.

The geodesic $\rho(\gamma)$ is shown in Figure 1.9. Note that $\rho(v_\gamma) = u_{\rho(\gamma)}$. We define $v_{\rho(\gamma)}$ similarly to above, but now we have $\rho(\gamma_+) = -[1; 3, 3, \dots]$. Thus, $v_{\rho(\gamma)}$ is where $\rho(\gamma)$ crosses $x = -1$, and $\rho(\gamma)$ crosses one triangle labeled R. So, now we know that part of the cutting sequence is $u_\gamma L^2 v_\gamma R^1 v_{\rho(\gamma)}$. Continuing this process will give the entire cutting sequence for geodesics that end at γ_+.

The cutting sequence between γ_- and $x = 0$: We use $\frac{-1}{\gamma_-}$ to convert to the above cases. The action $\frac{-1}{z}$ flips Ls to Rs and vice versa. However, we have already found the cutting sequence for $\frac{-1}{\gamma_-} = \gamma_+$ starts with L^2R^1. Thus, we now know that our example cutting sequence looks like $\dots L^1 R^2 u_\gamma L^2 v_\gamma R^1 \dots$, agreeing with the labeling in Figure 1.8. □

The cutting sequences corresponding to regular continued fractions allow us to describe geodesics on the corresponding modular surface. For example, some geodesics on the modular surface start and end at the same point. You can use cutting sequences to prove that these geodesics are identified with oriented geodesics on the hyperbolic surface where $\gamma_+ = \overline{[a_0; a_1, \dots, a_{2r}]}$ and $\gamma_- = -[0; \overline{a_{2r}, a_{2r-1}, \dots, a_1}]$, where the line means the digits repeat, as in repeating decimals.

We can also use ρ to help us find the invariant measure for $\overline{T}$. First, remember that ρ takes the set $[1, \infty) \times [-1, 0] \cup (-\infty, -1] \times [0, 1]$ to itself. Next, we define a map $f(x, y)$ from $[1, \infty) \times [-1, 0] \cup (\infty, -1] \times [0, 1]$ to $[0, 1]^2$ by

$$f(x,y) = \begin{cases} \left(\frac{1}{x}, -y\right) & \text{if } x \geq 1 \\ \left(\frac{-1}{x}, y\right) & \text{if } x \leq -1 \end{cases} = sign(x)\left(\frac{1}{x}, -y\right),$$

and $\overline{T} \circ f = f \circ \rho$. The map f is onto except for the line $(0, y)$ which has no preimage.

Proposition 3. *The invariant measure for ρ is $\iint h(x,y)dxdy = \iint \frac{dxdy}{(x-y)^2}$ (see for example Series [11, Section 3]).*

We will use the invariant measure of ρ to find the invariant measure of $\overline{T}$. To simplify notation, let $S = [1, \infty) \times [-1, 0] \cup (\infty, -1] \times [0, 1]$. The change of variables formula from multivariable calculus lets us use f^{-1} to find the invariant measure for $\overline{T}$.

Since f is not invertible, we have two cases: $f^{-1}(u, v) = (\frac{1}{u}, -v)$ and $f^{-1}(u, v) = (\frac{-1}{u}, v)$. In the first case, $f^{-1}(u, v) = (\frac{1}{u}, -v) = (x, y)$, so $x = \frac{1}{u}, y = -v$, and the Jacobian is $\frac{1}{u^2}$. Finally, the change of variables formula gives

$$\iint_S h(x,y)dxdy = \iint_{[0,1]^2} \frac{h(f^{-1}(u,v))}{u^2} dudv = \iint_{[0,1]^2} \frac{1}{u^2} \frac{dudv}{(\frac{1}{u} - (-v))^2} = \iint_{[0,1]^2} \frac{dudv}{(1+uv)^2}.$$

The other case gives the same result. Thus, the invariant measure for $\overline{T}$ is $\iint \frac{dxdy}{(1+xy)^2}$.

It is possible to directly verify that $\frac{dxdy}{(1+xy)^2}$ is the invariant measure for $\overline{T}$, as [3] does for the even continued fractions. However, the geometric construction allowed us to find the measure more easily.

1.6 Contributions

My contribution to the field is a cutting sequence representation of even continued fractions, as well as those with odd denominators [3]. This is joint work with Florin Boca. We translate from the cutting sequence for regular continued fractions to the cutting sequence for even continued fractions according to the insertion and singularization algorithm.

The cutting sequences of regular continued fractions and even continued fractions both use the Farey tessellation, but they have different underlying geometries as described in the introduction. (See references [3, 11] for more details). Each of the continued fraction expansions corresponds to a specific group and provides a description of something called *group equivalence*. The correspondence between the groups and continued fractions are another result that we get from the geometric construction.

1.6.1 Even continued fractions

We can also use cutting sequences to find the even continued fraction expansion of γ_+ and the extended even continued fraction expansion of γ_-. We will use the same last step as the regular continued fractions, that is, for some set $S_e \subset \mathbb{R}^2$, the map $f : S_e \to [0,1] \times [-1,1]$ is given by

$$f(x,y) = \begin{cases} \left(\frac{1}{x}, -y\right) & \text{if } x \geq 1 \\ \left(\frac{-1}{x}, y\right) & \text{if } x \leq -1 \end{cases} = sign(x)\left(\frac{1}{x}, -y\right),$$

and $\overline{T_e} \circ f = f \circ \rho_e$. We use $[0,1] \times [-1,1]$ for this map since $\overline{T_e}$ is defined on $[0,1] \times [-1,1]$. Now, we just need to find S_e, ρ_e, and a way to convert between cutting sequences and the even and extended even continued fraction expansions of γ_+ and γ_-. Note that since f is the same as in the regular continued fraction case, the multivariable calculus change of variables formula gives that the invariant measure for $\overline{T_e}$ is again $\iint \frac{dxdy}{(1+xy)^2}$.

Proposition 4. *Since we are still working with the Farey tessellation, the invariant measure for ρ_e is still $\iint \frac{dxdy}{(x-y)^2}$. Since we are using the same function f as in Proposition 3, the invariant measure for $\overline{T}_e$ is still $\iint \frac{dxdy}{(1+xy)^2}$.*

Again, it is possible to directly verify that $\iint \frac{dxdy}{(1+xy)^2}$ is the invariant measure for $\overline{T}_e$ (for example, Section 6.1 of [3]).

To find S_e, we can use $f^{-1}(x,y) = (\gamma_+, \gamma_-)$. Then the two cases are

$$(\gamma_+, \gamma_-) = \left(\frac{1}{x}, -y\right) \in [1, \infty) \times [-1, 1],$$
$$(\gamma_+, \gamma_-) = \left(\frac{-1}{x}, y\right) \in (-\infty, -1] \times [-1, 1].$$

That is, we consider the cases $\gamma_+ \geq 1$ and $\gamma_+ \leq -1$. In both cases, $\gamma_- \in [-1, 1]$.

Since these geodesics will no longer cross $x = 0$, the lines $x = 1$ and $x = -1$ will separate the part of γ corresponding to γ_+ and γ_-. In this case, γ will cross $x = 1$ when γ_+ is positive, and it will cross $x = -1$ when γ_+ is negative. To account for this change, u_γ is now the place γ crosses $x = \pm 1$ and v_γ is the next time γ crosses an arc connecting two integers. The cutting sequence between u_γ and v_γ will describe the first digit (ordered pair (a_i, e_i)) of the even continued fraction expansion of γ_+. This way, we move both u_γ and v_γ one letter to the right in the cutting sequence.

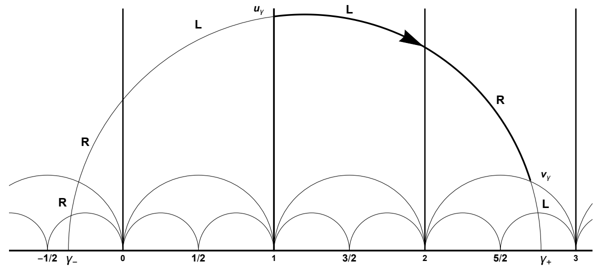

Figure 1.11. The geodesic γ with endpoints $\gamma_+ = \alpha$ and $\gamma_- = -1/\alpha$. Segments are labeled L or R based on how the geodesic crosses the Farey tessellation giving the sequence $R^2 L u_\gamma L R v_\gamma L$.

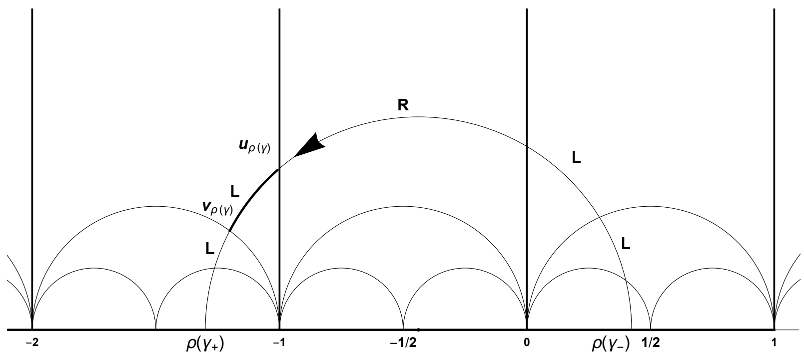

Figure 1.12. The geodesic $\rho_e(\gamma)$ with endpoints $\rho(\gamma_+)$ and $\rho_e(\gamma_-)$. Segments are labeled L or R based on how the geodesic crosses the Farey tessellation giving the sequence $RL^2 R u_{\rho(\gamma)} L v_{\rho(\gamma)} L$.

From equation (1.6), we know that

$$\alpha = [2; 1, 3, 3, \dots] = 2 + \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{4+z}}}} = [\![(2, +1); (2, -1)(2, -1)(2, -1)(4, +1), \dots]\!],$$

$$\frac{1}{\alpha} = [0; 2, 1, 3, 3, \dots] = \cfrac{1}{2 + \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{4+z}}}}} = \langle\!\langle (+1/2)(+1/2)(-1/2)(-1/2)(-1/4), \dots \rangle\!\rangle. \tag{1.8}$$

From section 1.5.2, we know that the cutting sequence for the geodesic with endpoints $\gamma_+ = \alpha$ and $\gamma_- = \frac{-1}{\alpha}$ contains $L^3 R^3 L^1 R^2 L^2 R^1 L^3 R^3$, where the imaginary axis is between $L^3 R^3 L^1 R^2$ and $L^2 R^1 L^3 R^3$. We can also see these strings in Figures 1.8 and 1.11, which show the geodesic from $\gamma_- = \frac{-1}{\alpha}$ to $\gamma_+ = \alpha$ along with its cutting sequence. We rewrite the cutting sequence to include the new u_γ and v_γ and get $L^3 R^3 L^1 R^2 L u_\gamma L R v_\gamma L^3 R^3$. As we will show below, the first 'digit' (ordered pair) of the even continued fraction expansion of $\gamma_=$ is determined by the portion of the cutting sequence between u_γ and v_γ. In the case of α, the first digit is $(2, +1)$; we see that this corresponds to the sequence LR.

When $\gamma_+ \in [2k, 2k+1)$, we know from the regular continued fraction case that the cutting sequence for γ is $\dots L u_\gamma L^{2k-1} R v_\gamma \dots$, so $L^{2k-1} R$ corresponds to the digit $(2k, +1)$. This matches with the example where LR corresponds to $(2, +1)$. When $\gamma_+ \in [2k-1, 2k)$, the cutting sequence is $\dots L u_\gamma L^{2k-2} R v_\gamma \dots$, and $L^{2k-2} R$ corresponds to the digit $(2, -1)$. Again, the case $\gamma_+ \leq -1$ is analogus, with the roles of L and R switched. Note that it is possible to get $L^0 R$ when $\gamma_+ \in [1, 2)$ and $R^0 L$ when $\gamma_+ \in (-2, -1]$, as in Figure 1.12.

As with the regular continued fractions, we define a map ρ_e that acts like $\bar{T}_e$ on (γ_+, γ_-). That is, ρ_e takes the first digit of γ_+ and moves it to the first digit of γ_-. Thus, the first digit of the even continued fraction expansion corresponds to the section of the cutting sequence between u_γ and v_γ. The second digit corresponds to the section of $\rho_e(\gamma)$ between $u_{\rho_e(\gamma)}$ and $v_{\rho_e(\gamma)}$, and we can continue this process to find the even continued fraction expansion of γ_+.

If

$$\gamma_+ = sign(\gamma_+)\left(a_1 + \cfrac{e_1}{a_2 + \cfrac{e_2}{a_3 + \cfrac{e_3}{\dots}}}\right) = sign(\gamma_+)[\![(a_1, e_1); (a_2, e_2)(a_3, e_3), \dots]\!],$$

$$\gamma_- = -sign(\gamma_+)\left(\cfrac{e_0}{a_0 + \cfrac{e_{-1}}{a_{-1} + \cfrac{e_{-2}}{a_{-1}} + \dots}}\right) = -sign(\gamma_+)\langle\!\langle (e_0/a_0, e_0)(e_{-1}/a_{-1})(e_{-2}/a_{-2}), \dots \rangle\!\rangle,$$

we define

$$\rho_e(x, y) = \left(\frac{1}{sign(\gamma_+)a_1 - x}, \frac{1}{sign(\gamma_+)a_1 - y}\right)$$

as in the regular continued fraction case. Now

$$\begin{aligned}
\rho_e(\gamma_+, \gamma_1) &= \left(\frac{1}{sign(\gamma_+)a_1 - x}, \frac{1}{sign(\gamma_+)a_1 - y}\right) \\
&= \left(\frac{1}{sign(\gamma_+)a_1 - sign(\gamma_+)\left(a_1 + \cfrac{e_1}{a_2 + \cfrac{e_2}{\dots}}\right)}, \frac{1}{sign(\gamma_+)a_1 + sign(\gamma_+)\left(\cfrac{e_0}{a_0 + \dots}\right)}\right) \\
&= \left(-sign(\gamma_+)e_1\left(a_2 + \frac{e_2}{\dots}\right), \frac{sign(\gamma_+)}{a_1 + \cfrac{e_0}{a_0 + \dots}}\right) \\
&= \left(-sign(\gamma_+)e_1\left(a_2 + \frac{e_2}{\dots}\right), sign(\gamma_+)e_1 \frac{e_1}{a_1 + \cfrac{e_0}{a_0 + \dots}}\right).
\end{aligned}$$

When $e_1 = +1$, we multiply by -1 as in the regular continued fraction case, but do not when $e_1 = -1$. Again, we find that $\rho_e(v_\gamma) = u_{\rho_e(\gamma)}$.

Returning to our example $\gamma_+ = \alpha, \gamma_- = -1/\alpha$, we see that

$$\rho_e(\gamma_+, \gamma_-) = ([\![(2, -1); (2, -1)(2, -1)(4, +1), \dots]\!], \langle\!\langle (+1/2)(+1/2)(+1/2)(-1/2)(-1/2), \dots \rangle\!\rangle).$$

The cutting sequence of $\rho_e(\gamma)$ contains $L^3R^3L^1R^2Lu_\gamma LRu_{\rho_e(\gamma)}Lv_{\rho_e(\gamma)}L^2R^3$. Thus, $u_{\rho_e(\gamma)}Lv_{\rho_e(\gamma)}$ corresponds to $(2, -1)$ as desired. Since we cannot (easily) tell from the cutting sequence when there's a hidden L^0 or R^0, summarizing the rules for reading the even continued fraction expansion of γ_+ from the cutting sequence is more complicated.

Theorem 5.
- *For a geodesic with $\gamma_+ > 1$ and $-1 < \gamma_- < 1$, we find:*
  - $\gamma_+ = [\![(2k_1, +1); (2k_2, e_2)\dots]\!], \gamma_- = -\langle\!\langle (2k_0, e_0)(2k_{-1}, e_{-1})\dots\rangle\!\rangle_e$ *has cutting sequence* $\dots Lu_\gamma L^{2k_1-1}Rv_\gamma R\dots$
  - $\gamma_+ = [\![(2k_1, -1); (2k_2, e_2)\dots]\!], \gamma_- = -\langle\!\langle (2k_0, e_0)(2k_{-1}, e_{-1})\dots\rangle\!\rangle_e$ *has cutting sequence* $\dots Lu_\gamma L^{2k_1-2}Rv_\gamma R\dots$ *when* $2k_1 > 2$ *and* $\dots Lu_\gamma L^0 Rv_\gamma R\dots = \dots Lu_\gamma Rv_\gamma R\dots$ *when* $2k_1 = 2$.

We use $\rho_e(\gamma)$ to find the next digit string corresponding to $(2k_2, e_2)$.

- *For a geodesic with $\gamma_+ < -1$ and $-1 < \gamma_- < 1$, we find:*
  - $\gamma_+ = -[\![(2k_1, +1); (2k_2, e_2)\dots]\!], \gamma_- = \langle\!\langle(2k_0, e_0)(2k_{-1}, e_{-1})\dots\rangle\!\rangle_e$ *has cutting sequence* $\dots Ru_\gamma R^{2k_1-1} Lv_\gamma L\dots$
  - $\gamma_+ = -[\![(2k_1, -1); (2k_2, e_2)\dots]\!], \gamma_- = \langle\!\langle(2k_0, e_0)(2k_{-1}, e_{-1})\dots\rangle\!\rangle_e$ *has cutting sequence* $\dots Ru_\gamma R^{2k_1-2} Lv_\gamma L\dots$ *when* $2k_1 > 2$ *and* $\dots Ru_\gamma Lv_\gamma L\dots$ *when* $2k_1 = 2$.

  We use $\rho_e(\gamma)$ to find the next digit string corresponding to $(2k_2, e_2)$.

- *To find the first digit of the extended even continued expansion of γ_-, we apply ρ_e^{-1} and read the cutting sequence between $u_{\rho_e^{-1}(\gamma)}$ and $v_{\rho_e^{-1}(\gamma)}$ using the same rules. We continue this process to get the rest of the digits.*

1.7 Ideas for future study

There are many online resources with basic patterns with continued fractions. The website [8] has a long list of patterns in the digits of regular continued fraction expansions. The article "Christiaan Huygens' Planetarium" [2] proves several number theoretic facts about regular continued fractions before describing how Huygens used continued fractions to construct a mechanical planetarium.

If you are interested in learning more about geometric constructions of continued fractions, it would be good to take number theory, dynamics, and differential geometry courses. The textbook *Dynamics Done with Your Bare Hands* [4] introduces several ways of studying dynamical systems. The first chapter is the most similar to this paper and is available online [5]. On the continued fraction side, Masarotto [9] carefully describes many types of continued fraction expansions in her masters thesis.

My paper "Coding of geodesics on some modular surfaces and applications to odd and even continued fractions" with Florin Boca [3] discusses the cutting sequences corresponding to even continued fractions in greater depth. This paper also describes the cutting sequences corresponding to the continued fractions with odd denominators which relate to a different subgroup of determinant 1 matrices. For both the even and the odd continued fractions, the paper describes some of the number theoretic results, as well as using the continued fraction expansions to calculate lengths of geodesics on the surfaces.

Once you are familiar with tangent spaces and some topology, I suggest reading Series's paper [11] that was outlined in Section 1.5.2. This paper carefully describes the regular continued fraction coding using ideas from differential geometry and topology.

For graduate-level sources, Katok and Ugarcovini [7] and Adler and Flatto [1] wrote articles on many connections between hyperbolic geometry and continued fractions. Adler and Flatto in particular discuss the connections between the hyperbolic plane and matrix groups. Finally, Einsiedler and Ward's book *Ergodic theory with a view towards Number Theory* [6] covers many of related topics from dynamical systems.

1.8 Acknowledgements

I am grateful to the SMP community for their constant support through graduate school, including while writing this paper. I would also like to thank the referees for their suggestions that greatly improved this paper. This research was completed while I was a graduate student at the University of Illinois at Urbana-Champagin.

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