The subject of *Diophantine Approximation* concerns approximating real numbers with rational numbers. One way to do this is with *Farey fractions*–which we can generate by adding "wrong." However, there is a much simpler way to generate Farey fractions. One goal is to show these two methods give the same sequences.

1 First definition of Farey sequences

Definition 1. *The* N^{th} Farey sequence *is the list of all fractions, written from smallest to largest, between* 0 *and* 1 *where the denominator is less than or equal to* N *when written as a reduced fraction. We write this sequence as* $\mathcal{F}_N$.

The first three Farey sequences are:

$$\mathcal{F}_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\},$$
$$\mathcal{F}_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\},$$
$$\mathcal{F}_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

1. Write out $\mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6$. Before reading beyond this first question, make as many observations about the Farey sequences as you can–you do not need to prove any of them. How can you find the elements of $\mathcal{F}_6$ just knowing $\mathcal{F}_5$? Are there any patterns? List all the observations you make and patterns you discover.

Solution:

$$\mathcal{F}_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$
$$\mathcal{F}_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$$
$$\mathcal{F}_6 = \left\{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}\right\}$$

Possible observations:

- Always add $\frac{1}{n}$ after 0 and $\frac{n-1}{n}$ before 1. Since $\frac{1}{6}$ and $\frac{5}{6}$ are the only fractions in reduced form with denominator 6, this is how you get from $\mathcal{F}_5$ to $\mathcal{F}_6$.
- The sequence of denominators is symmetric.
- For $\mathcal{F}_5$, we added $\frac{2}{5}$ between $\frac{1}{3}$ and $\frac{1}{2}$ and $\frac{3}{5}$ between $\frac{1}{2}$ and $\frac{2}{3}$, and $2 + 3 = 5$.

How many numbers are in $\mathcal{F}_N$? There is not a nice formula for this number, but we can find a formula using the *Euler phi function*, $\phi(n)$, which is the number of positive integers less than or equal to n and do not share any common factors with n. For example, $\phi(9) = 6$ since there are exactly six numbers in the range from 1 to 9 that do not share any common factors with 9: $1, 2, 4, 5, 7, 8$.

2. Find $\phi(n)$ for n from 1 to 10.

Solution:

$$
\begin{aligned}
\phi(1) = 1, \phi(2) &= 1, \phi(3) = 2, \phi(4) &= 2, \phi(5) = 4, \\
\phi(6) = 2, \phi(7) &= 6, \phi(8) = 4, \phi(9) &= 6, \phi(10) = 4.
\end{aligned}
$$

As an interesting aside, it is not known whether every possible value of $\phi(n)$ occurs at least twice. It is known that $\phi(n)$ is never 3.

Lemma 1. *For a positive integer $N \geq 2$, the number of elements of $\mathcal{F}_N$ with denominator N is equal to $\phi(N)$.*

3. Why is this lemma true? Use this to write a formula for the number of elements of $\mathcal{F}_N$ in terms of the Euler ϕ function.

Solution: Since the elements of $\mathcal{F}_N$ are always in reduced form, we only include those were the numerator and denominator do not share any common factors. Since we include all of these fractions, we have one fraction for each which is the number of positive integers less than or equal to N and do not share any common factors with N.

Then number of elements of $\mathcal{F}_N$ is then the number of elements of $\mathcal{F}_{N-1} + \phi(N)$. There are 2 elements in $\mathcal{F}_1$. Then $2+\phi(2)$ in $\mathcal{F}_2$, $2+\phi(2)+\phi(3)$ in $\mathcal{F}_3, \ldots, 2+\phi(2)+\phi(3)+\cdots+\phi(N)$ in $\mathcal{F}_N$.