The subject of *Diophantine Approximation* concerns approximating real numbers with rational numbers. One way to do this is with *Farey fractions*–which we can generate by adding "wrong."

# 1 Definition of Farey sequences

One way to generate the Farey sequence is using a table. In the first row, write $\frac{0}{1}$ and $\frac{1}{1}$.

To form the second row, copy the first row. Then insert $\frac{0+1}{1+1}$ between $\frac{0}{1}$ and $\frac{1}{1}$.

To form the $n^{th}$ row, copy the $(n-1)^{st}$ row. Then for each $\frac{a}{b}, \frac{c}{d}$ in the $(n-1)$ row, if $b+d \leq n$, insert $\frac{a+c}{b+d}$ between $\frac{a}{b}$ and $\frac{c}{d}$.

1. Here are the first four rows. Notice that the Here are the first four rows. Notice that the we did not include any values between $\frac{1}{3}$ and $\frac{1}{2}$ in row 4, since $2+3>4$.

   Fill in rows 5 and 6

| | | | | | | |
|---|---|---|---|---|---|---|
| $\frac{0}{1}$ | | | | | | $\frac{1}{1}$ |
| $\frac{0}{1}$ | | | $\frac{1}{2}$ | | | $\frac{1}{1}$ |
| $\frac{0}{1}$ | | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{2}{3}$ | | $\frac{1}{1}$ |
| $\frac{0}{1}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{2}{3}$ | $\frac{3}{4}$ | $\frac{1}{1}$ |

**Solution:**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{0}{1}$ | | | | | | | | | | | | $\frac{1}{1}$ |
| $\frac{0}{1}$ | | | | | $\frac{1}{2}$ | | | | | | | $\frac{1}{1}$ |
| $\frac{0}{1}$ | | | $\frac{1}{3}$ | | $\frac{1}{2}$ | | $\frac{2}{3}$ | | | | | $\frac{1}{1}$ |
| $\frac{0}{1}$ | | | $\frac{1}{4}$ | $\frac{1}{3}$ | | $\frac{1}{2}$ | $\frac{3}{5}$ | $\frac{2}{3}$ | $\frac{3}{4}$ | | | $\frac{1}{1}$ |
| $\frac{0}{1}$ | | $\frac{1}{5}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{2}{5}$ | $\frac{1}{2}$ | $\frac{3}{5}$ | $\frac{2}{3}$ | $\frac{3}{4}$ | $\frac{4}{5}$ | | $\frac{1}{1}$ |
| $\frac{0}{1}$ | $\frac{1}{6}$ | $\frac{1}{5}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{2}{5}$ | $\frac{1}{2}$ | $\frac{3}{5}$ | $\frac{2}{3}$ | $\frac{3}{4}$ | $\frac{4}{5}$ | $\frac{5}{6}$ | $\frac{1}{1}$ |

**Theorem 1.** *If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in the $n^{th}$ row of the table, with $\frac{a}{b}$ to the left of $\frac{c}{d}$, then $bc - ad = 1$.*

2. The goal of the this question is to prove the theorem. We will do this using a technique called *induction*. First, we will show this is true for the first row. Then, we will show that *if the theorem is true for the $(n-1)^{st}$ row*, it is also true for the $n^{th}$ row.

   (a) Show that the theorem is true for the first row.

   (b) Assume that for every pair of consecutive $\frac{a}{b}$ and $\frac{c}{d}$ in the $(n-1)^{st}$ row of the table, $ad - bc = 1$. Now, for a specific pair $\frac{a}{b}, \frac{c}{d}$ in the $(n-1)^{st}$ row, there are two cases for the $n^{th}$ row. What are they?

   (c) For each of the two cases, show that the theorem is true.

**Solution:**

(a) There are only two elements of the first row. $1(1) - 0(1) = 1$.

(b) If $b + d \leq n$, then we insert $\frac{a+c}{b+d}$ in between $\frac{a}{b}$ and $\frac{c}{d}$. Otherwise, $\frac{a}{b}$ and $\frac{c}{d}$ are also consecutive in the $n^{th}$ row.

(c) If $b + d \leq n$, then we need to check the pairs $\frac{a}{b}, \frac{a+b}{b+d}$ and $\frac{a+b}{b+d}, \frac{c}{d}$. Now

$$b(a + c) - a(b + d) = bc - ad,$$

which is 1 by assumption. Similarly,

$$c(b + d) - d(a + b) = bc - ad.$$

In the other case, $bc - ad = 1$ by assumption.

Here are two corollaries and a theorem that you do not have to prove:

**Corollary 2.** *Every $\frac{a}{b}$ in the table is in reduced form.*

**Corollary 3.** *The fractions in each row are listed in order from smallest to largest.*

**Theorem 4.** *If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in* any *row, and $\frac{r}{s}$ is any rational number where $\frac{a}{b} < \frac{r}{s} < \frac{c}{d}$, then the smallest possible denominator for $\frac{r}{s}$ is $b+d$ and $\frac{a+c}{b+d}$ is the only one with this denominator.*

3. Use the induction and Theorem 4 to show:

**Theorem 5.** *If $0 \leq m \leq n$ and $m$ and $n$ have no common factors, the fraction $\frac{m}{n}$ is in the $n^{th}$ row of the table. Therefore, this table is equivalent to the definition: The $N^{th}$* Farey sequence *is the list of all fractions, written from smallest to largest, between 0 and 1 where the denominator is less than or equal to $N$ when written as a reduced fraction.*

(a) Show that the theorem is true for the first row.

(b) The rule for adding an element to the table is: for each $\frac{a}{b}, \frac{c}{d}$ in the $(n-1)$ row, if $b+d \leq n$, insert $\frac{a+c}{b+d}$ between $\frac{a}{b}$ and $\frac{c}{d}$. We call $\frac{a+c}{b+d}$ the *mediant of $\frac{a}{b}$ and $\frac{c}{d}$*.
Assume Theorem 5 is true for the $(n-1)^{st}$ row. That is, the $(n-1)^{st}$ row is a list of all fractions, written from smallest to largest, between 0 and 1 where the denominator is less than or equal to $n-1$ when written as a reduced fraction. How can this simplify the rule for adding an element to the table?

(c) If $b+d > n$, $\frac{a+c}{b+d}$ is not in the $n^{th}$ row of the table. Now we just need to prove that every fraction $\frac{m}{n}$ where $m$ and $n$ have no common factors is in the $n^{th}$ row of the table. Do this.

**Solution:**

(a) The first row of the table is $\{\frac{0}{1}, \frac{1}{1}\}$, which is all fractions between 0 and 1 with denominator 1.

(b) Assume that the $(n-1)^{st}$ row is a list of all fractions, written from smallest to largest, between 0 and 1 where the denominator is less than or equal to $n-1$ when written as a reduced fraction.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be consecutive fractions in the $(n-1)^{st}$ row. If $b+d < n$, it is already in the table by assumption. So we only need $b+d = n$.

(c) Since $\frac{m}{n}$ is not in the $(n-1)^{st}$ row, it is between two consecutive Farey fractions $\frac{a}{b}$ and $\frac{c}{d}$. By Theorem 4, the smallest possible denominator for fractions between $\frac{a}{b}$ and $\frac{c}{d}$ is $b+d$ and $\frac{a+c}{b+d}$ is the only one with this denominator. Since $\frac{a+c}{b+d}$ is not in the $(n-1)^{st}$ row, $b+d \geq n$. Since $\frac{a}{b} \frac{m}{n} < \frac{c}{d}$, $n \geq b+d$. Thus, $b+d = n$ and $\frac{m}{n} = \frac{a+c}{b+d}$. Thus, $\frac{m}{n}$ is in the $n^{th}$ row.

4. Let $\frac{a}{b}$ and $\frac{c}{d}$ be the fractions immediately to the left and right of $\frac{1}{2}$ in the $n^{th}$ Farey sequence. Prove that $b = d$ is the greatest odd integer less than or equal to $n$. Also prove $a + c = b$.

**Solution:** This is true for the second row $\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$.

Assume this is true for the $(n-1)^{st}$ row. If $n-1$ is odd, then $\frac{a}{b} \leq 12\frac{c}{d}$ and $b = d = n-1$. By Theorem 4, $b+2 = d+2 = n+1$ is the smallest possible denominator for fractions between $\frac{a}{b}$ and $\frac{1}{2}$ and between $\frac{1}{2}$ and $\frac{c}{d}$. Thus, the fractions $\frac{a+1}{b+2}$ and $\frac{c+1}{d+2}$ are added on either side of $\frac{1}{2}$ in the $(n+1)^s t$ row. We also have that $a + c = n - 1$, and $(a + 1) + (c + 1) = n + 1$.

If $n-1$ is even, then $\frac{a}{b} \leq 12\frac{c}{d}$ and $b = d = n-2$. By Theorem 4, $b+2 = d+2 = n$ is the smallest possible denominator for fractions between $\frac{a}{b}$ and $\frac{1}{2}$ and between $\frac{1}{2}$ and $\frac{c}{d}$. Thus, the fractions $\frac{a+1}{b+2}$ and $\frac{c+1}{d+2}$ are added on either side of $\frac{1}{2}$ in the $n^{th}$ row. We also have that $a + c = n - 1$, and $(a + 1) + (c + 1) = n + 1$.

## 2 Approximating rational numbers

Here's another result that does not seem to immediately use Farey fractions:

**Theorem 6** (Dirichlet's theorem, 1842)**.** *For all $x \in \mathbb{R}$ and $Q \in \mathbb{N}$, there exists $p/q \in \mathbb{Q}$ with $q \leq Q$ such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{qQ}. \tag{1}$$

*In particular, if $x$ is irrational, there exist infinitely many rational numbers $p/q$ such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{2}$$

5. Show that if $\frac{p}{q} \neq \frac{5}{7}$, then
$$\left| \frac{5}{7} - \frac{p}{q} \right| \geq \frac{1}{7q}.$$
Formulate and prove a similar result where $\frac{5}{7}$ is replaced with an arbitrary rational number $\frac{r}{s}$. [*Hint: What do we know if $\frac{5}{7}$ and $\frac{p}{q}$ are consecutive in some Farey sequence? What do we know if they are not?*]

**Solution:** If $\frac{5}{7}$ and $\frac{p}{q}$ are consecutive in some Farey sequence, then $|5p - 7q| = 1$. Then

$$\left|\frac{5}{7} - \frac{p}{q}\right| = \left|\frac{5p - 7q}{7q}\right| = \frac{1}{7q}.$$

If $\frac{5}{7}$ and $\frac{p}{q}$ are not consecutive in some Farey sequence, then there is some other $\frac{a}{b}$ between $\frac{5}{7}$ and $\frac{p}{q}$, so the difference is greater.

We can generalize the Farey sequence by removing the restriction "between 0 and 1." For example, the Farey sequence of order 2 becomes

$$\ldots, \frac{-3}{1}, \frac{-5}{2}, \frac{-2}{1}, \frac{-3}{2}, \frac{-1}{1}, \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \ldots$$

The same theorems are true in this case, except the number of elements is always infinite.

We can approximate an irrational number $x$ using the following algorithm:

(i) Start with the interval $[\frac{n}{1}, \frac{n+1}{1}]$ where $n < x < n+1$

(ii) Compute the mediant of the endpoints of the interval.

(iii) Of the two new intervals, keep the one containing $x$. Go back to *(ii)* and repeat.

This algorithm provides a sequence of mediants converging to $x$.

6. Here are 3 rational approximations of $\pi$, found using another method:

$$3, \frac{22}{7}, \frac{333}{106}$$

Compare this with the sequence you get by applying the Farey fraction algorithm 10 times. With $\pi$ as a reference point, illustrate the sequence of mediants obtained on a number line.

**Solution:** We start with $\frac{3}{1} < \pi < \frac{4}{1}$. The first mediant approximation is then $\frac{7}{2}$. Next, we get $\frac{3+7}{1+2} = \frac{10}{3}$, then $\frac{3+10}{1+3} = \frac{13}{4}$, $\frac{3+13}{1+4} = \frac{16}{5}$, $\frac{3+16}{1+5} = \frac{19}{6}$, $\frac{22}{7}$, $\frac{22+19}{7+6} = \frac{41}{13}$, $\frac{22+41}{7+13} = \frac{63}{20}$, $\frac{22+63}{7+20} = \frac{85}{27}$, $\frac{85+22}{7+27} = \frac{107}{34}$.
The mediant approximation is less than $\pi$ until it equals $\frac{22}{7}$, which takes 7 approximations (if you count 3, then it is greater than $\pi$. It turns out this will be true until the mediant approximation is $\frac{333}{106}$.

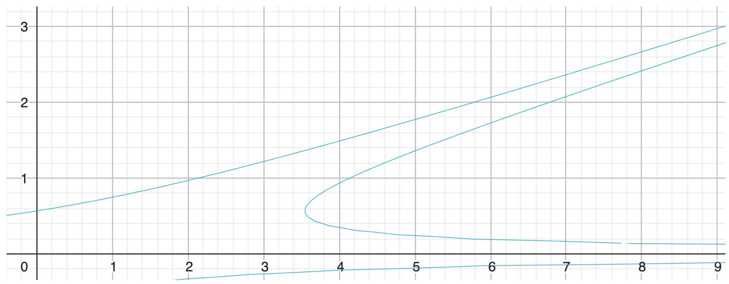
# 3 Some geometry

A *lattice point* is a point in the $xy$-plane where $x$ and $y$ are both integers.

7. Plot the line $x = \alpha y$ and the set of solutions to the inequality $|\alpha - \frac{x}{y}| < \frac{1}{y^2}$ in the $xy$-plane. What does Dirichlet's theorem say about lattice points in this "funnel?" How does the picture change when $\alpha$ is rational or irrational? Find all several lattice points $(x, y)$ in the $xy$-plane such that $|\pi - \frac{x}{y}| < \frac{1}{y^2}$.

**Solution:** Dirichlet's theorem says there are infinitely many lattice points in the funnel when $\alpha$ is irrational. If $\alpha$ is rational, these points may be on the line $x = \alpha y$.

Graph for $\alpha = \pi$




8. For each rational number $\frac{p}{q}$, draw a circle in the plane of radius $\frac{1}{2q^2}$ with center $(\frac{p}{q}, \frac{1}{2q^2})$. Observe that the circles corresponding to consecutive terms in a Farey sequence are tangent, ie, touch at exactly one point. Explain why.

**Solution:** The circles intersect at exactly one point if and only if the distance between the centers is equal to the sum of the radii of the circles.

Let

$$d^2 = \left(\frac{c}{d} - \frac{a}{b}\right)^2 + \left(\frac{1}{2d^2} - \frac{1}{2b^2}\right)^2 = \frac{1}{b^2d^2} + \frac{1}{4d^4} - \frac{1}{2b^2d^2} + \frac{1}{4b^4} = \frac{1}{4d^4} + \frac{1}{2b^2d^2} + \frac{1}{4b^4},$$

the square of the distance between the centers of the circles.

Let $s = \frac{1}{2d^2} + \frac{1}{2b^2}$ be the sum of the radii of the circles. So $s^2 = \frac{1}{4d^4} + \frac{1}{2b^2d^2} + \frac{1}{4b^4} = d^2$.

Since the sum of the radii is equal to the distance between the centers, there is exactly one point on both circles.