The subject of *Diophantine Approximation* concerns approximating real numbers with rational numbers. One way to do this is with *Farey fractions*–which we can generate by adding "wrong."

1 Definition of Farey sequences

One way to generate the Farey sequence is using a table. In the first row, write $\frac{0}{1}$ and $\frac{1}{1}$.

To form the second row, copy the first row. Then insert $\frac{0+1}{1+1}$ between $\frac{0}{1}$ and $\frac{1}{1}$.

To form the n^{th} row, copy the $(n-1)^{st}$ row. Then for each $\frac{a}{b}, \frac{c}{d}$ in the $(n-1)$ row, if $b+d \leq n$, insert $\frac{a+c}{b+d}$ between $\frac{a}{b}$ and $\frac{c}{d}$.

1. Here are the first four rows. Notice that the Here are the first four rows. Notice that the we did not include any values between $\frac{1}{3}$ and $\frac{1}{2}$ in row 4, since $2+3>4$.

   Fill in rows 5 and 6

$\frac{0}{1}$												$\frac{1}{1}$
$\frac{0}{1}$						$\frac{1}{2}$						$\frac{1}{1}$
$\frac{0}{1}$				$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$				$\frac{1}{1}$
$\frac{0}{1}$			$\frac{1}{4}$	$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$	$\frac{3}{4}$			$\frac{1}{1}$

Theorem 1. *If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in the n^{th} row of the table, with $\frac{a}{b}$ to the left of $\frac{c}{d}$, then $bc - ad = 1$.*

2. The goal of the this question is to prove the theorem. We will do this using a technique called *induction*. First, we will show this is true for the first row. Then, we will show that *if the theorem is true for the* $(n-1)^{st}$ *row*, it is also true for the n^{th} row.
   (a) Show that the theorem is true for the first row.
   (b) Assume that for every pair of consecutive $\frac{a}{b}$ and $\frac{c}{d}$ in the $(n-1)^{st}$ row of the table, $ad - bc = 1$. Now, for a specific pair $\frac{a}{b}, \frac{c}{d}$ in the $(n-1)^{st}$ row, there are two cases for the n^{th} row. What are they?
   (c) For each of the two cases, show that the theorem is true.

Here are two corollaries and a theorem that you do not have to prove:

Corollary 2. *Every $\frac{a}{b}$ in the table is in reduced form.*

Corollary 3. *The fractions in each row are listed in order from smallest to largest.*

Theorem 4. *If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in* any *row, and $\frac{r}{s}$ is any rational number where $\frac{a}{b} < \frac{r}{s} < \frac{c}{d}$, then the smallest possible denominator for $\frac{r}{s}$ is $b+d$ and $\frac{a+c}{b+d}$ is the only one with this denominator.*

3. Use the induction and Theorem 4 to show:

   Theorem 5. *If $0 \leq m \leq n$ and m and n have no common factors, the fraction $\frac{m}{n}$ is in the n^{th} row of the table. Therefore, this table is equivalent to the definition: The N^{th}* Farey sequence *is the list of all fractions, written from smallest to largest, between 0 and 1 where the denominator is less than or equal to N when written as a reduced fraction.*

   (a) Show that the theorem is true for the first row.

   (b) The rule for adding an element to the table is: for each $\frac{a}{b}, \frac{c}{d}$ in the $(n-1)$ row, if $b+d \leq n$, insert $\frac{a+c}{b+d}$ between $\frac{a}{b}$ and $\frac{c}{d}$. We call $\frac{a+c}{b+d}$ the *mediant of $\frac{a}{b}$ and $\frac{c}{d}$.*
   Assume Theorem 5 is true for the $(n-1)^{st}$ row. That is, the $(n-1)^{st}$ row is a list of all fractions, written from smallest to largest, between 0 and 1 where the denominator is less than or equal to $n-1$ when written as a reduced fraction. How can this simplify the rule for adding an element to the table?

   (c) If $b+d > n$, $\frac{a+c}{b+d}$ is not in the n^{th} row of the table. Now we just need to prove that every fraction $\frac{m}{n}$ where m and n have no common factors is in the n^{th} row of the table. Do this.

4. Let $\frac{a}{b}$ and $\frac{c}{d}$ be the fractions immediately to the left and right of $\frac{1}{2}$ in the n^{th} Farey sequence. Prove that $b = d$ is the greatest odd integer less than or equal to n. Also prove $a + c = b$.

2 Approximating rational numbers

Here's another result that does not seem to immediately use Farey fractions:

Theorem 6 (Dirichlet's theorem, 1842)**.** *For all $x \in \mathbb{R}$ and $Q \in \mathbb{N}$, there exists $p/q \in \mathbb{Q}$ with $q \leq Q$ such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{qQ}. \tag{1}$$

In particular, if x is irrational, there exist infinitely many rational numbers p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{2}$$

5. Show that if $\frac{p}{q} \neq \frac{5}{7}$, then

$$\left| \frac{5}{7} - \frac{p}{q} \right| \geq \frac{1}{7q}.$$

Formulate and prove a similar result where $\frac{5}{7}$ is replaced with an arbitrary rational number $\frac{r}{s}$. [*Hint: What do we know if $\frac{5}{7}$ and $\frac{p}{q}$ are consecutive in some Farey sequence? What do we know if they are not?*]

We can generalize the Farey sequence by removing the restriction "between 0 and 1." For example, the Farey sequence of order 2 becomes

$$\ldots, \frac{-3}{1}, \frac{-5}{2}, \frac{-2}{1}, \frac{-3}{2}, \frac{-1}{1}, \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \ldots$$

The same theorems are true in this case, except the number of elements is always infinite.

We can approximate an irrational number x using the following algorithm:

(i) Start with the interval $[\frac{n}{1}, \frac{n+1}{1}]$ where $n < x < n+1$

(ii) Compute the mediant of the endpoints of the interval.

(iii) Of the two new intervals, keep the one containing x. Go back to *(ii)* and repeat.

This algorithm provides a sequence of mediants converging to x.

6. Here are 3 rational approximations of π, found using another method:

$$3, \frac{22}{7}, \frac{333}{106}$$

Compare this with the sequence you get by applying the Farey fraction algorithm 10 times. With π as a reference point, illustrate the sequence of mediants obtained on a number line.

3 Some geometry

A *lattice point* is a point in the xy-plane where x and y are both integers.

7. Plot the line $x = \alpha y$ and the set of solutions to the inequality $|\alpha - \frac{x}{y}| < \frac{1}{y^2}$ in the xy-plane. What does Dirichlet's theorem say about lattice points in this "funnel?" How does the picture change when α is rational or irrational? Find all several lattice points (x, y) in the xy-plane such that $|\pi - \frac{x}{y}| < \frac{1}{y^2}$.

8. For each rational number $\frac{p}{q}$, draw a circle in the plane of radius $\frac{1}{2q^2}$ with center $(\frac{p}{q}, \frac{1}{2q^2})$. Observe that the circles corresponding to consecutive terms in a Farey sequence are tangent, ie, touch at exactly one point. Explain why.